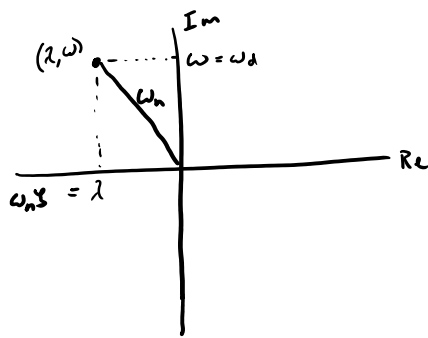


Sticky note: list three review topics ranked in order 1 to 3 that you'd like me to review

- HW #9 Due Friday (2 HW is exempt)
- Lab 5 Due Monday (not accepting late!)
- Exam: W Mar. 18 3:30-5:30 PM

# Eigen values, damping ratio, natural frequency



$$x_1(t) = A e^{\lambda t} \sin(\omega t + \phi)$$

$\uparrow$  damped natural frequency

$$\omega_d = \omega_n \sqrt{1 - \zeta^2}$$

$\uparrow$  natural frequency       $\uparrow$  damping ratio

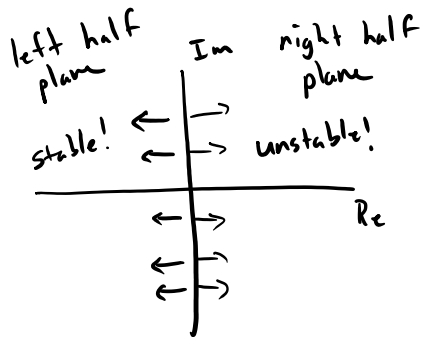
$$\omega_d^2 = \omega_n^2 (1 - \zeta^2)$$

$$\omega_d^2 = \omega_n^2 - \omega_n^2 \zeta^2$$

$$\omega_d^2 + \omega_n^2 \zeta^2 = \omega_n^2$$



$$x(t) = A e^{\omega_n \zeta t} \sin(\omega_d t + \phi)$$



$$\bar{x}(t) = \bar{X} e^{st}$$

Eigenvectors :  $\bar{X}$

- each eigenvector is associated with an eigenvalue
- each eigenvector can have real and imaginary parts
- eigenvectors provide two things:
  - relative magnitude of the states
  - phase shift between any oscillating states

$$[sI - A]\bar{X} = 0$$

plug in  $s_1, \dots, s_n$  and solve for the associated eigenvector  $\bar{X}_1, \dots, \bar{X}_n$  for  $n$  states.

Example

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad s_{1,2} = \lambda \pm \omega i$$

$$\overset{sI - A}{\begin{bmatrix} \lambda + \omega i - a_{11} & -a_{12} \\ -a_{21} & \lambda + \omega i - a_{22} \end{bmatrix}} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(\lambda + \omega i - a_{11})X_1 - a_{12}X_2 = 0$$

$$-a_{21}X_1 + (\lambda + \omega i - a_{22})X_2 = 0$$

$$X_2 = \frac{\lambda + \omega i - a_{11}}{a_{12}} X_1$$

$$-a_{21}X_1 + \frac{(\lambda + \omega i - a_{22})(\lambda + \omega i - a_{11})}{a_{12}} X_1 = 0$$

$$X_1 = 0 \quad \text{non-unique solution}$$

Pick  $X_1 = 1$   
 $X_2 = \frac{\lambda + \omega i - a_{11}}{a_{12}}$  } this is one of many solutions!

$$s_1 = \lambda + \omega i \quad \bar{X}_1 = \begin{bmatrix} 1 \\ \frac{\lambda + \omega i - a_{11}}{a_{12}} \end{bmatrix} = \begin{bmatrix} 1 \\ \Delta + \mathcal{L}i \end{bmatrix} \quad \Delta = \frac{\lambda - a_{11}}{a_{12}}$$

$$\mathcal{L} = \frac{\omega}{a_{12}}$$

$$s_2 = \lambda - \omega i \quad \bar{X}_2 = \begin{bmatrix} 1 \\ \frac{\lambda - \omega i - a_{11}}{a_{12}} \end{bmatrix} = \begin{bmatrix} 1 \\ \Delta - \mathcal{L}i \end{bmatrix}$$

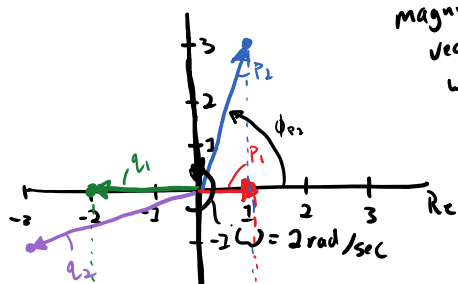
### Interpreting Eigenvectors

Ex  $n=4$  so 4 eigenvalues total

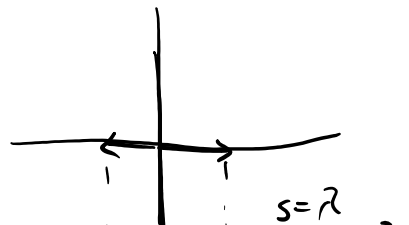
$$\bar{x} = \begin{bmatrix} p_1 \\ p_2 \\ q_1 \\ q_2 \end{bmatrix} \quad s_{1,2} = -3 \pm 2i \leftarrow \text{two of the eigenvalues}$$

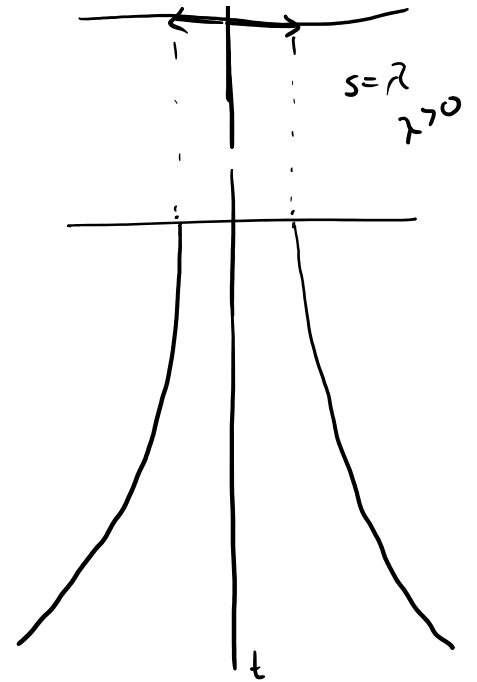
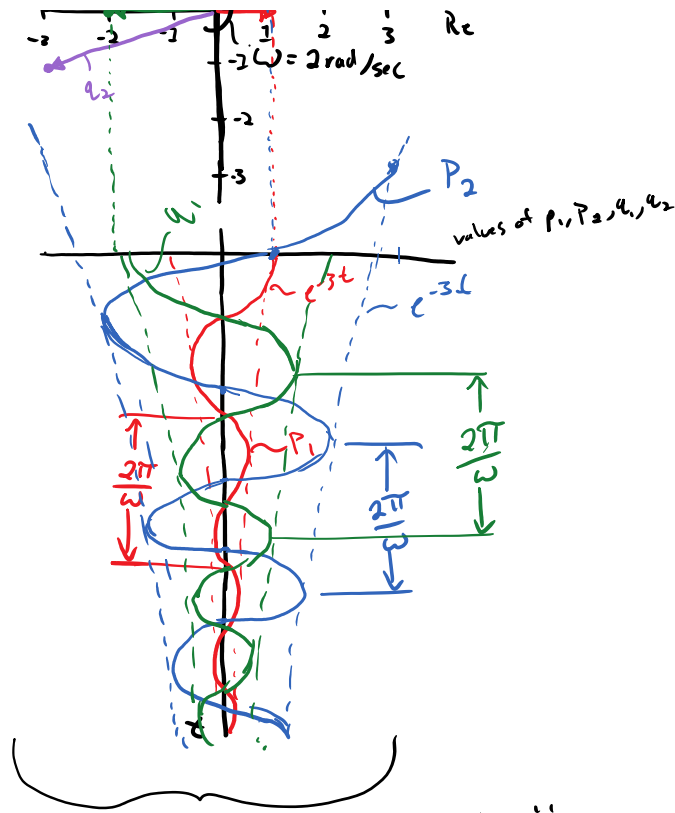
$$\bar{X}_1 = \begin{bmatrix} 1 \\ 1+3i \\ -2 \\ -3-i \end{bmatrix} \quad \bar{X}_2 = \begin{bmatrix} 1 \\ 1-3i \\ -2 \\ -3+i \end{bmatrix} \rightarrow \begin{matrix} p_1 \\ p_2 \\ q_1 \\ q_2 \end{matrix}$$

Plot the components on the imaginary plane.



magnitudes of the vectors decrease with  $e^{-3t}$





a single "mode of motion"

Eigen value/Eigenvector pairs represent one of the modes.  
The superposition of all modes gives the total motion.

$$\bar{X}(t) = \underbrace{\sum_{i=1}^{n_r} d_i \bar{X}_i e^{\lambda_i t}}_{\text{non-oscillatory modes}} + \underbrace{\sum_{j=1}^{n_i} d_j \bar{X}_j \sin(\omega_j t + \phi_j)}_{\text{oscillatory modes}} + \underbrace{\sum_{k=1}^{n_{ri}} d_k \bar{X}_k e^{\lambda_k t} \sin(\omega_k t + \phi_k)}_{\text{decaying/growing oscillatory modes}}$$

$n_r$ : number of real eigenvalues

$n_i$ : number of complex pairs with  $\lambda = 0$

$n_{ri}$ : number of complex pairs with  $\lambda \neq 0$

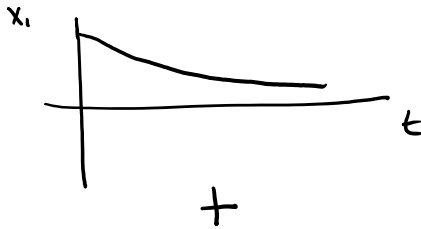
$$n_r + 2n_i + 2n_{ri} = n \quad \hookrightarrow \# \text{ of eigenvalues, } \# \text{ states}$$

$d_{i,j,k}$ : modal participation factors } depend on initial conditions,  $\bar{X}(0)$

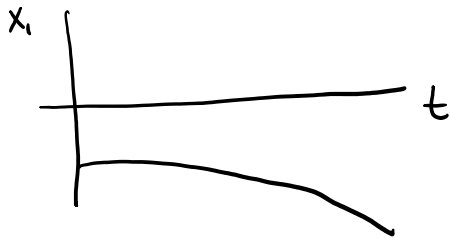
$d_{i,j,k}$ : modal participation factors  
 $\phi_{j,k}$ : phase shift between oscillatory modes  
 $\bar{\Gamma}_{i,j,k}$ : mag & phase shift among states in that mode

} depends on initial conditions,  $\bar{X}(0)$

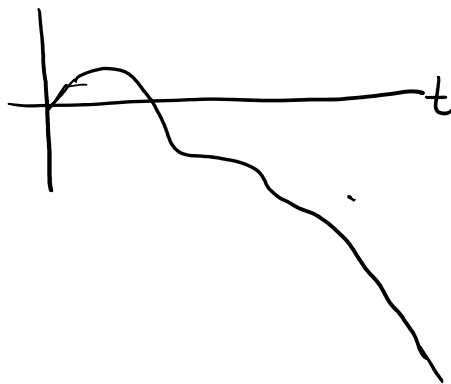
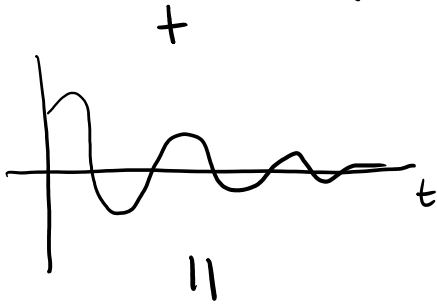
mode 1



mode 2

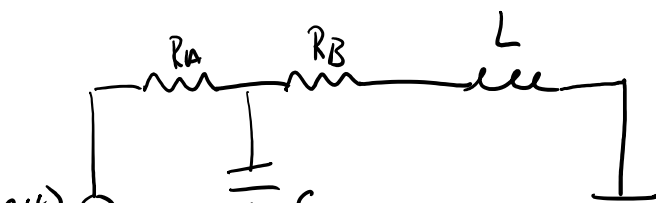


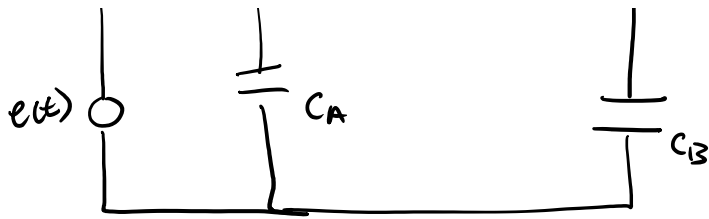
mode 3



if any mode is unstable, the total motion is unstable!!

### Example





3x3

$$\begin{bmatrix} \dot{q}_4 \\ \dot{P}_7 \\ \dot{q}_8 \end{bmatrix} = \begin{bmatrix} -\frac{1}{R_A C_A} & -\frac{1}{L} & 0 \\ \frac{1}{C_A} & -\frac{R_B}{L} & -\frac{1}{C_B} \\ 0 & \frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} q_4 \\ P_7 \\ q_8 \end{bmatrix} + \begin{bmatrix} 1/R_A \\ 0 \\ 0 \end{bmatrix} [e(t)]$$

$$R_A = 1 \Omega$$

$$C_A = 10 \text{ mF}$$

$$L = 1.5 \text{ mH}$$

$$R_B = 2 \Omega$$

$$C_B = 6 \mu\text{F}$$