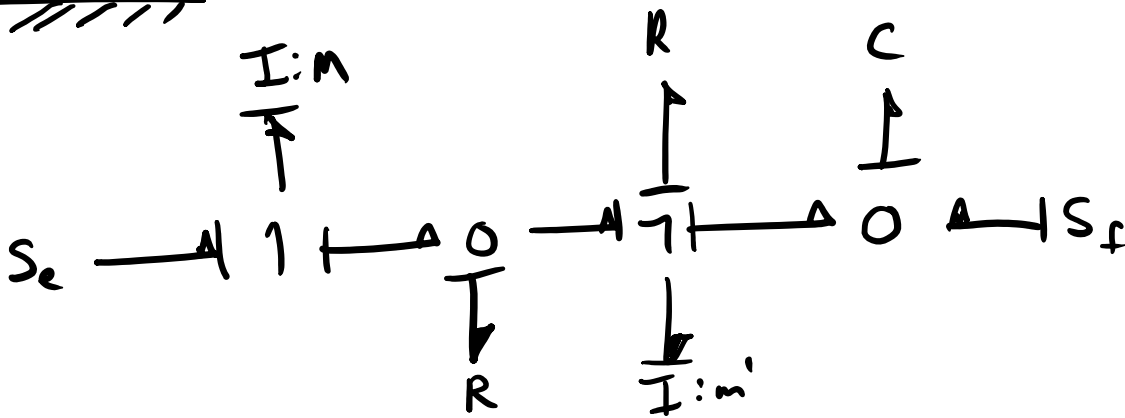
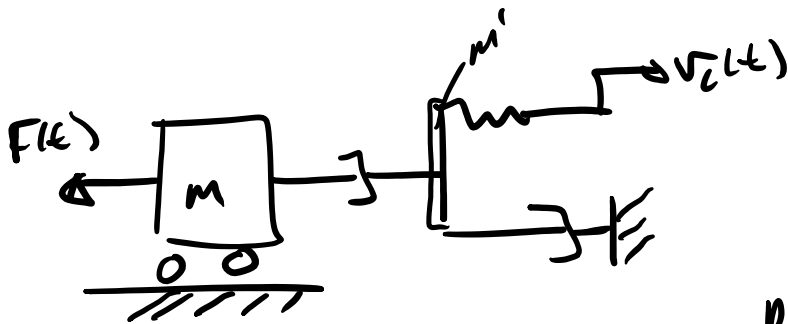


# Avoiding Algebraic Loops

Add  $I$  or  $C$  to  
get rid of loops.  
but carefully!



# Derivative Causality

$\dot{\bar{x}} = f(\bar{x}, t) \Rightarrow$  explicit: all derivatives are on the left hand side

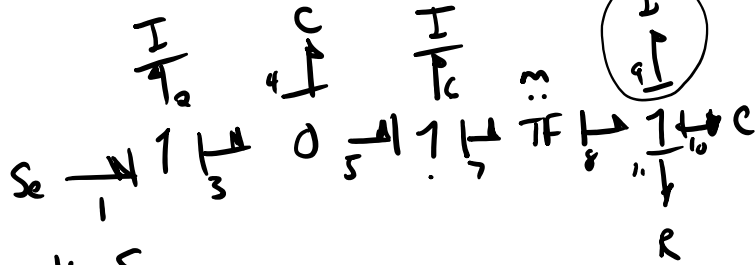
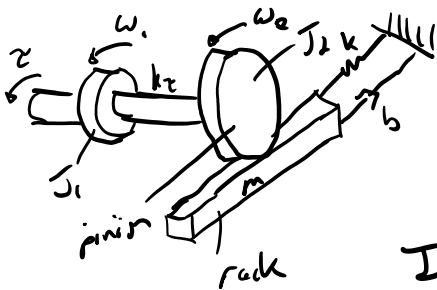
Derivative Causality causes equations to be in this form:

$\dot{\bar{x}} = f(\dot{\bar{x}}, \bar{x}, t) \Rightarrow$  implicit: derivatives on right and left side of equations

$\dot{\bar{x}} = M\dot{\bar{x}} + N\bar{x} \rightarrow$  linear is possible to convert to explicit

Non-linear  $\rightarrow$  may be possible to convert to explicit if not you have a DAE Algebraic Equations

## Example



Input:  $S_e$

States:  $P_2, q_4, P_6, q_{10}$

not  $P_9!$

$$P_9 = I_1 f_1 = I_1 m f_7 = I_1 m \frac{P_6}{I_C}$$

$$P_9 = \frac{I_1}{I_C} m P_6 \Rightarrow \overset{\text{non-state}}{P_9} = \frac{I_1}{I_C} m \overset{\text{state}}{P_6}$$

$$\textcircled{1} \dot{P}_2 = e_2 = S_{e1} - e_3 = S_{e1} - \frac{q_4}{c_4}$$

$$\textcircled{3} \dot{P}_6 = e_6 = e_5 - e_7 = \frac{q_4}{c_4} - m e_8$$

$$\textcircled{2} \dot{q}_4 = f_4 = f_2 - f_c = \frac{P_2}{I} - \frac{P_6}{I}$$

$$P_6 = \frac{q_4}{c_4} - m \left( e_9 + \frac{q_{10}}{c_{10}} + R_{11} f_{11} \right)$$

$\perp 2 \quad \perp 6$

$$\dot{P}_6 = \frac{q_4}{c_4} - m \left( \dot{P}_1 + \frac{q_{10}}{c_{10}} + R_{11} m \frac{P_6}{I_6} \right)$$

$$\dot{P}_6 = \frac{q_4}{c_4} - m \left( \frac{I_9}{I_6} m \dot{P}_6 + \frac{q_{10}}{c_{10}} + \frac{R_{11} m}{I_6} P_6 \right)$$

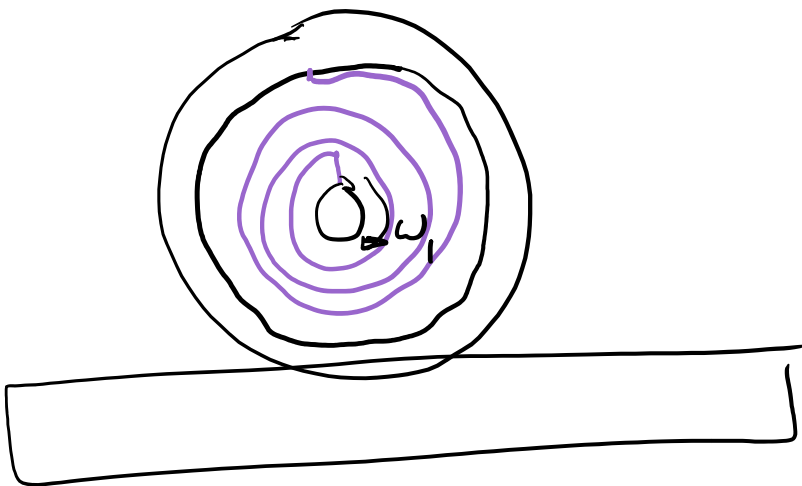
$$\dot{P}_6 = \frac{\frac{q_4}{c_4} - \frac{m q_{10}}{c_{10}} - \frac{R_{11} m^2}{I_6} P_6}{1 + \frac{I_9}{I_6} m^2}$$

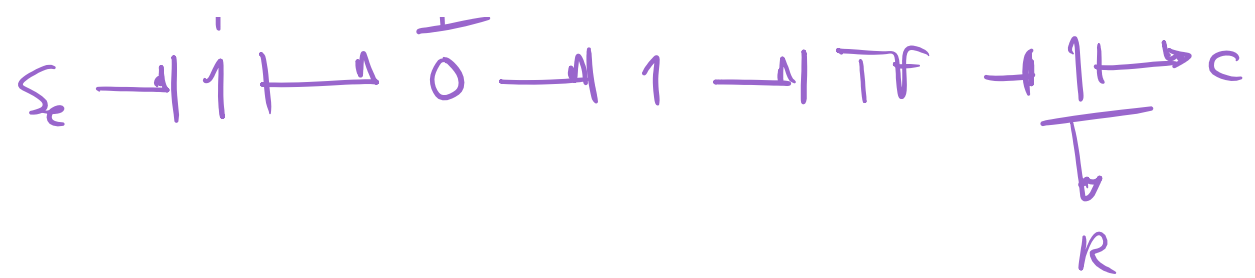
$\Rightarrow$  explicit form of the equation!

④  $\dot{q}_{10} = \dot{f}_{10} = m f_7 = m \frac{P_6}{I_6}$

If  $C$  is derivative causality:

$$q = C e$$





# Linear System Solutions

$$\dot{\bar{x}} = A\bar{x} + B\bar{u}$$

$$\bar{y} = C\bar{x} + D\bar{u}$$

A: state matrix

B: input matrix

C: output matrix

D: feedforward matrix

desire state  
↓ trajectory

$$\bar{x}(t) = \int_{t_0}^{t_f} \dot{\bar{x}} dt$$

⇒ for linear systems there is analytical solution to this integral

If  $\bar{u} = 0 \Rightarrow$  find homogeneous solution

Turns out that a solution for  $\dot{\bar{x}} = A\bar{x}$

$$\left. \begin{aligned} \bar{x}(t) &= \bar{X} e^{st} \\ \dot{\bar{x}}(t) &= \bar{X} s e^{st} \end{aligned} \right\}$$

↑ plugin

$$\bar{X} s e^{st} = A \bar{X} e^{st}$$

$$\bar{X} e^{st} [sI - A] = 0$$

↑ identity matrix

$$\bar{X} = 0 \Rightarrow \text{trivial solution}$$

$e^{st} = 0 \Rightarrow$  impossible

$\bar{X}[sI - A] = 0 \Rightarrow$  non-trivial solution

$\det|sI - A| = 0 \Rightarrow$  to find  $\underline{s}$  (unique solution)

For  $2 \times 2$  remember:

$$\det \begin{vmatrix} s - a_{11} & -a_{12} \\ -a_{21} & s - a_{22} \end{vmatrix} = 0 \Rightarrow (s - a_{11})(s - a_{22}) + a_{12}a_{21} = 0$$

$s^2 - (a_{11} + a_{22})s + a_{11}a_{22} - a_{12}a_{21} = 0$   
s? quadratic formula      characteristic equation of A

quadratic polynomial in s

$$s_{1,2} = \frac{a_{11} + a_{22}}{2} \pm \frac{\sqrt{(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})}}{2}$$

$\underbrace{\hspace{10em}}_{s}$  are the eigenvalues

depending on entries of A:

$s_{1,2} = \lambda \pm \omega i$   $\sqrt{-1}$  real and imaginary pair

$s_1 = \lambda_1, s_2 = \lambda_2$  two real eigenvalues

$s_{1,2} = \pm \omega i$  imaginary pair

$$s_{1,2} = \pm \omega i$$

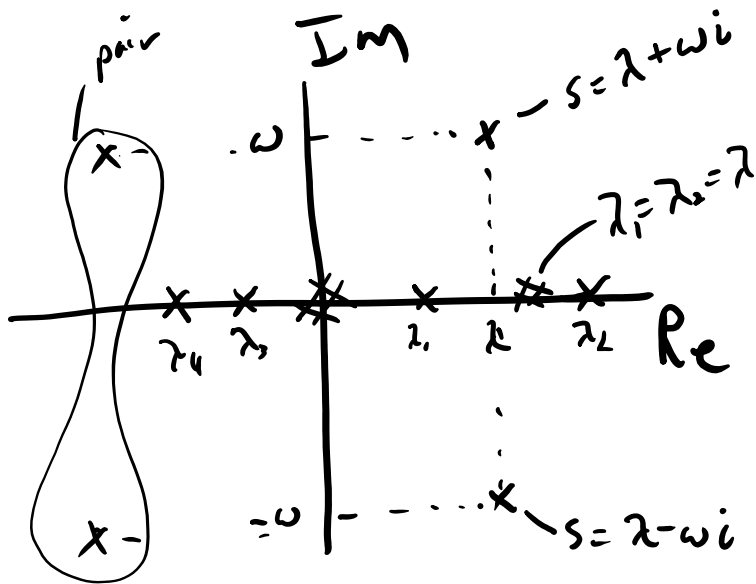
imaginary pair

$$s_{1,2} = 0$$

$$s_{1,2} = \lambda$$

1 real, same value for each

For an  $n \times n$   $A$  matrix there are  $n$  eigenvalues.  
And a  $n^{\text{th}}$  order polynomial characteristic equation. We find the roots of this polynomial.  
↳ eigenvalues



Now we know  $s = \lambda \pm \omega i$

$$\bar{X}(t) = \bar{X} e^{st} = \bar{X} e^{(\lambda \pm \omega i)t}$$

↳ for scalar  $\bar{X} = X$

$$X(t) = Y e^{\lambda t} \sin(\omega t + \phi)$$